Journal of Mathematical Chemistry Vol. 36, No. 1, May 2004 (© 2004)

The algebraic structure of physical quantities

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Received 11 December 2003; revised 10 March 2004

We present a general algebraic basis for arbitrary systems of units such as those used in physical sciences, engineering, and economics. Physical quantities are represented as q-numbers: an ordered pair $u = \{\mathrm{u}, \mathrm{label}_{\mathrm{u}}\}$, that is, $u \in \mathbf{q} = \mathbf{X} \times \mathbf{W}_B$. The algebraic structure of the infinite sets of labels that represent the "units" has been established: such sets $\mathbf{W}_B$ are infinite Abelian multiplicative groups with a finite basis. $\mathbf{W}_B$ is solvable as it admits a tower of Abelian subgroups. Extensions to include the possibility of rational powers of labels have been included, as well as the addition of named labels. Named labels are an essential feature of all practical systems of units. Furthermore, $\mathbf{q}$ is an Abelian multiplicative group, and it is not a ring. $\mathbf{q}$ admits decomposition into one-dimensional normed vector spaces over the field $\mathbf{X}$ among members with equivalent labels. These properties lead naturally to the concept of well-posed relations, and to Buckingham's theorem of dimensional analysis. Finally, a connection is made with a Group Ring structure and an interpretation in terms of the observable properties of physico-chemical systems is given.

KEY WORDS: algebraic structure, physical quantities

AMS subject classification: 20K99, 92E99, 00A73, 00A79

1. Introduction

The physical sciences and engineering disciplines are quantitative sciences, and as such, significant numerical computation and theoretical work is carried out using physical quantities. However, these quantities are not pure numbers, but rather more complex objects that include a label. For example, when we measure an area of a sphere of radius 1 cm, we obtain $4\pi\ \mathrm{cm}^2$, while if we measure the energy of a particle of 10^{-3} kg traveling at $5 \times 10^2\ \mathrm{km\,s}^{-1}$ in free space, we obtain, 250 J. To distinguish these quantities from pure numbers ($\mathbf{C}$, the complex numbers), we will call them q-numbers, and these will be denoted herein with italics.

The algebraic structure of the labels of physical quantities has been hinted at in many books on dimensional analysis [1–3]. Starting with the classic work by Bridgman [1], most of these books concentrate on the applications to the derivation of dimensionally correct equations in various fields of physics and

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engineering. Langhaar [2] even includes a chapter called "Algebraic Theory of Dimensional Analysis", but the word "group" is nowhere to be found in this chapter. The mathematical foundation taken by engineers, chemists and physicists has depended on the concept of dimensional homogeneity without actually developing the abstract algebra [3]. De Jong [4] has presented the utility of dimensional analysis in the field of economics and his book includes a translation of an article written more than 40 years ago by Quade [5], who presented a full abstract algebraic description of dimensional analysis using an Abelian group construction based on vector spaces. However, Quade's approach is restrictive, and the important extensions to names and rational exponents were not considered.

For example, consider the expression for the root mean squares displacement from the origin of a Brownian particle executing a random walk in unbounded three-dimensional space:

$$x_{\mathrm{rms}} = \sqrt{\langle x^2 \rangle} = \sqrt{6Dt},$$

D is the diffusion coefficient, with units $\mathrm{m}^2\ \mathrm{s}^{-1}$, and t is the time, with units in s (m is the abbreviation of meter, and s, that of second). In practice, we do not worry about whether we do this computation by first multiplying out all the numerical quantities, yielding $6Dt\ \mathrm{m}^2$ and then taking the square root, or whether we take the square root of each quantity separately, multiply, and treat the units as follows: $\mathrm{m}\ \mathrm{s}^{-1/2}\ \mathrm{s}^{1/2} = \mathrm{m}$. The basic question can then be stated: does there exist an abstract algebraic structure wherein the latter manipulations are naturally defined? Quade's answer is no, you must never take the square root of a basic unit; only integral powers are allowed. This answer is unsatisfactory – much more general structures can actually be defined.

In this work we obtain an algebraic foundation for systems of units as used in practice in the natural sciences and economics. Such a system should be large enough to include, for example, square roots of basic scales, for they do appear in calculations and in certain systems of units discussed further below. A further example is the activity coefficient of an ionic species in a polyelectrolyte solution. The logarithm of the activity, in the limiting case of small ionic strength, is proportional to the square root of ionic strength, which has units, $\mathrm{mole}^{1/2}\,\mathrm{kg}^{-1/2}$. Another interesting development that requires the consideration of more general exponents is the fractional calculus [6]. The application to diffusion and kinetics is discussed in the recent very accessible article by Sokolov et al. [7]

In this paper, we present a flexible algebraic structure that can accommodate practically any system of units, including scaling and convenience names for labels. In addition, we make contact with a modern algebraic structure that can be interpreted as containing the description of all properties of all possible physical systems. The presentation is directed principally to non-mathematicians, in the language of mathematicians.

2. Definitions

2.1. *q-number*

A quantity u is a q-number iff u is the ordered pair, $u = \{\mathrm{u}, \mathrm{label_u}\}$, where $\mathrm{u} \in \mathbf{C}$, the complex numbers, and $\mathrm{label_u} \in \mathbf{W}$. Often u is restricted to the real numbers, $\mathbf{R}$.

The set $\mathbf{W}$ is the set of "words" that we can use as labels, commonly referred to as "units". For the above definition to have meaning, we need to specify the set $\mathbf{W}$ and any operations among the members of this set. However, for convenience, we will define the basic operation first and thereby obtain some compact nomenclature to fully specify the set $\mathbf{W}$. Note that we will use "=" to signify "assignment", and "= =" to signify an equality relation.

2.2. *Multiplication of labels*

We define a multiplication operator "." between labels as follows. For any $\mathrm{label}_j \in \mathbf{W}$:

(1) $\mathrm{label_1} \cdot \mathrm{label_2} = \mathrm{label_1 label_2} \in \mathbf{W}$, "Closure"

(2) $(\mathrm{label_1} \cdot \mathrm{label_2}) \cdot \mathrm{label_3} = \mathrm{label_1} \cdot (\mathrm{label_2} \cdot \mathrm{label_3}) = \mathrm{label_1 label_2 label_3}$, "Associativity"

(3) There exists a unique "unit" label, $\mathbf{1} \in \mathbf{W}$, such that:

$$\mathbf{1} \cdot \mathrm{label} = \mathrm{label} \cdot \mathbf{1} = \mathrm{label}, \quad \text{“Neutral element”}$$

(4) For every $\mathrm{label} \in \mathbf{W}$, there exists a unique inverse label, label^{-1} such that,

$$\mathrm{label}^{-1} \cdot \mathrm{label} = \mathrm{label} \cdot \mathrm{label}^{-1} = \mathbf{1}, \quad \text{“Inverse”}$$

For convenience, we can also represent the inverse as the label, 1/label.

(5) $\mathrm{label_1} \cdot \mathrm{label_2} = \mathrm{label_2} \cdot \mathrm{label_1}$, "Commutativity"

Thus, the structure $\{\mathbf{W}, .\}$ is an Abelian group [8,9]. There are no other operations defined on $\mathbf{W}$. In particular, there is no "addition" operation on labels, so $\mathbf{W}$ is not a ring, much less a field. We "can't add apples to oranges", is the vernacular expression of this property.

2.3. *Powers of labels*

The power "n" of a label, label^n, for $n > 0 \in \mathbf{Z}$, integers, is defined as an iterated multiplication:

$\text{label}^n = \text{label}\cdot\text{label}\cdots\text{label}\cdot\text{label} = \text{label label}\cdots\text{label label}$, n times.

By the inverse property, there exists an inverse for this object, which is the iterated multiplication of the inverse label, label^{-n}.

From the iterated multiplication definition it is readily observed that the normal rules for exponents are satisfied for powers of labels, viz., $\text{label}^n \cdot\text{label}^m = \text{label}^{n+m}$, where n and m are any integers. Thus, we also define the zero power for any label $\in \mathbf{W}$ as the unit label:

$$\text{label}^0 = \mathbf{1}.$$

2.4. *Rational powers*

Consider the object: $\text{label}^{1/n}$, $n > 0 \in \mathbf{Z}$. For $n = 1$, we just have label. We would like this object to belong to the set $\mathbf{W}$ for any n. Then, this power of label will essentially have to be a generator for $n > 1$, and $\mathbf{W}$ will not be finitely generated. If we multiplicatively iterate this label m times, then we produce $(\text{label}^{1/n})^m = \text{label}^{m/n}$. Thus the natural definition leads us to the consideration of an arbitrary rational exponent, and the iteration of $\text{label}^{1/n}$, n times, yields label.

Furthermore, we extend the definition of the inverse label for rational powers of a label:

$$(\text{label}^a)^{-1} = \text{label}^{-a},$$

so that the rules of exponents are satisfied over the rationals, $\mathbf{Q}$.

This construction can be clarified by considering the logarithm map log: $\mathbf{W} \to \mathbf{Q}$, where $\log[\text{label}^a] = a$, where a belongs to the commutative ring of fractions $\mathbf{Q}$. Since the elements of $\mathbf{Q}$ are defined as the equivalence classes of fractions [9], the logarithm map can be taken as bijective and invertible. Thus, the construction of a ring of fractions [9] carries over into the exponents of the labels in $\mathbf{W}$.

So far, the description of $\mathbf{W}$ is quite general, and it admits many types of realizations. We will be interested in specific realizations that have a finite basis, $\mathbf{B}$. The specification of the basis set allows for the explicit construction of the elements of the free Abelian group $\mathbf{W}_B$.

2.5. *Basis set and the free groups* $\mathbf{W}_B$

In actual systems of units, all labels are obtained from a small set of basis labels. These labels represent scales for various physical quantities, and thus can be chosen in many ways, generating different, but equivalent systems of units.

Thus, to keep the situation general, we will use a generic representation for the basis, $\mathbf{B}_n = \{l_1, \ldots, l_n\}$. Then, we can at once construct the set $\mathbf{W}_B$:

$$\mathbf{W}_B = \{l, l(a_1, \ldots, a_n) = \prod_{k=1}^{n} l_k^{a_k};\ \ a_k \in \mathbf{Q}, l_k \in \mathbf{B}_n\}.$$

This set is countably infinite because any label is essentially a map to an n-tuple of rational numbers: $l = l(a_1, \ldots, a_n)$. The set $\mathbf{W}_B$ is generated by the infinite set of words, $\{l_1^{1/k}, \ldots, l_n^{1/k}, |k| = 1, 2, 3, \ldots\}$, which contains all “roots” of the labels and their inverses. Then, it is evident that the explicit construction given above produces reduced labels (there are no redundant sequences containing the product of a generator and its inverse). Thus, the set $\mathbf{W}_B$ is is the free group generated from the basis B.

Definition. The *rank* of a group $\mathbf{W}_B$ is the cardinality of the basis set, $\mathbf{B}$.

With these definitions, the $\mathbf{W}_B$ are free Abelian groups of rank n. This, of course, is inconvenient. There are very long labels, in fact infinitely long labels, in the group $\mathbf{W}_B$. This inconvenience will be addressed later by allowing named labels to stand for a reduced label. Furthermore, there is the inconvenience of the magnitude of q-numbers, which ranges over many orders of magnitude in physical problems. The use of reduced labels forces us to carry large magnitudes of powers of 10 in the numeric representation of a q-number. This inconvenience will also be addressed by generalizing the group $\mathbf{W}_B$ to include scaled labels. A free group $\mathbf{W}_B$ is one were there is no scaling and no named labels.

Examples. The MKS system of units: $G = \{\text{kilogram, meter, second}\}$. The cgs system of units: $G = \{\text{gram, cm, second}\}$. These are systems of rank 3.

We must show that $\mathbf{W}_B$ constructed this way, combined with the label multiplication operation is an Abelian group.

Proof.

(1) *Closure*: the product of any two labels is in the set:

$$\text{label} \cdot \text{label}' = l^{a+a'} l^{b+b'} \cdots l^{e+e'} = l^{a''} l^{b''} \cdots l^{e''} \in \mathbf{W}_B$$

because the rationals are closed with respect to addition.

(2) *Associativity*: follows directly from the associativity of label multiplication.

(3) *Neutral element*: the label with all $a_k = 0$ is the unit label.

(4) *Inverse*: For any label not the unit label $= l^a l^b \cdots l^e$, label$^{-1} = l^{-a} l^{-b} \cdots l^{-e}$, whereas $\mathbf{1}^{-1} = \mathbf{1}$.

(5) *Commutativity*: follows directly from the commutativity of label multiplication. □

Lemma. Let $\mathbf{W}_k$ be the group $\mathbf{W}_B$ of rank k. Then the groups $\mathbf{W}_1, \ldots, \mathbf{W}_n$ form an Abelian tower.

Proof. First note that $\mathbf{W}_k \subset \mathbf{W}_{k+1}$, for $l(a_1, \ldots, a_k, 0) \in \mathbf{W}_{k+1}$, $l(a_1, \ldots, a_k, 0) = l(a_1, \ldots, a_k) \in \mathbf{W}_k$.

Since multiplication is commutative, then the subgroups are normal:

$l\mathbf{W}_k l^{-1} = \mathbf{W}_k$ for $l \in \mathbf{W}_{k+1}$. Now we need to show that the factor groups $\mathbf{W}_{k+1}/\mathbf{W}_k$ are Abelian. The factor group is the group of cosets of $\mathbf{W}_k$ with respect to $\mathbf{W}_{k+1}$. The cosets are sets of the form $l\mathbf{W}_k$. Then $(l_1\mathbf{W}_k)(l_2\mathbf{W}_k) = (l_2\mathbf{W}_k)(l_1\mathbf{W}_k)$ for $l_1, l_2 \in \mathbf{W}_{k+1}$, because all label multiplications are commutative. Thus the factor groups are Abelian. □

Remark. We can easily write down an expression for the cosets in our case, because the groups are infinite. The coset $l(a_1, \ldots, a_{k+1})\mathbf{W}_k = \{l, l(c_1, \ldots, c_k, a_{k+1}), c_j \in \mathbf{Q}\}$, contains all labels in $\mathbf{W}_{k+1}$ which have the a_{k+1} power of the $(k+1)$th basis element.

Remark. The Abelian tower can be extended to include the trivial subgroup containing only the unit label, since the unit label to any power is again the unit label. Thus, the group $\mathbf{W}_B$ is *solvable*.

Remark. Not all members of the tower are useful for even a restricted set of physical units. A group of labels must contain sufficient generators to specify at least, energy. This requires scales for mass, length, and time. Thus, the minimum rank that contains energy labels is 3. However, the rank 1 system containing only the generators for a length scale, is a useful system for simple mensuration.

2.6. *Cyclic subgroups*

Consider the set, $\mathbf{C}_k = \{l, l = l_k^j,\ j \in \mathbf{Z}\}$. This is clearly a cyclic group, of infinite period, generated by l_k. $\mathbf{C}_k$ is a subgroup all $\mathbf{W}_m$ with $m \geqslant k$. Furthermore, the product group $\prod_{j=1}^{k} \mathbf{C}_j$ is a subgroup of $\mathbf{W}_m$ with $m \geqslant k$. The product groups are not cyclic, but are Abelian. However, the product groups are not normal subgroups of $\mathbf{W}_n$, and form a non-Abelian tower in $\mathbf{W}_n$. This property arises because the exponents in $\mathbf{W}_n$ can range over the rationals and thus are not limited to integers. The groups $\mathbf{W}_n$ contain cyclic subgroups, but are not cyclic themselves.

$\mathbf{W}_B$ can also be properly characterized by noting that is a set of "pure monomials" of a Group Ring [10]. However, we emphasize that the polynomials in this Group Ring do not correspond to physical quantities, but can be given a related interpretation. We elaborate on this in section 4.

3. Extension to scaled labels and derived labels

As mentioned previously, the free groups are inconvenient for ordinary use. In practice, we use many names for labels that recur often. Let us describe them.

3.1. Scaled labels and derived labels

Scaling: It is convenient to enlarge our labels by considering multiplication by a *non-zero* real number, r.

Definition. Multiplication by a scalar: $r \cdot \text{label} = \text{label} \cdot r = r\text{label}$ for $r \neq 0 \in \mathbf{S} \subset \mathbf{R}$ and $\text{label} \in \mathbf{W}_B$. In addition, $r' \cdot (r \cdot \text{label}) = r' \cdot r\text{label} = r''\text{label}$ and $r, r'r'' \in \mathbf{S}$. $\mathbf{S}$ is an Abelian multiplicative subgroup of $\mathbf{R}$. In practice, $\mathbf{S} \subset \mathbf{Q}$.

Labels containing a real number r are called scaled labels. Typically, powers of 10 are chosen for the values of r, and new names are invented for such labels. For example, $\mu\text{m} = 10^{-6}$ m, where m is meter. Not all possible scaled labels have such names. We will denote a name by an identity mapping

Thus, $\mu\text{m} = \mathbf{name}[10^{-6}\ \text{m}] = 10^{-6}$ m. The name is unique. The existence of scaled labels implies that the representation of a q-number is not unique. Thus, the q-number u can be represented with a possibly infinite number of scaled labels, for a non-zero scaling r.

$$\mathbf{name}[r\ \text{label}] = r\ \text{label}.$$
$$u = \{\text{u},\ \text{label}\} = \{\text{u}/r, r\ \text{label}\} = \{\text{u}/r,\ \mathbf{name}[r\ \text{label}]\}.$$

Thus, scaling induces an equivalence relation in the q-numbers.

Derived labels: Products of labels can be also be given a unique name, as in the case of scaled labels. For example, a typical unit for viscosity, the Poise, is a scaled label:

$$\text{Poise} = \mathbf{name}[10^{-1}\,\text{kg m}^{-1}\,\text{s}^{-1}] = 10^{-1}\,\text{kg m}^{-1}\,\text{s}^{-1},$$

where kg, m, s are basis labels representing the kilogram, the meter, and the second, respectively. In practice, we just write the name of the derived label, say, Poise.

3.2. The generalized group $\mathbf{W}(B, S)$ and the names extension

The introduction of scaled labels implies that we must enlarge our group so that it will no longer be a free group. First we enlarge to include just scaled labels. Since we have added a new Abelian multiplicative group, **S**, we have a new structure.

$$\mathbf{W}(\mathbf{B}_n, \mathbf{S}) = \{l, l(a_1, \ldots, a_n; s) = r \prod_{k=1}^{n} l_k^{a_k}; a_k \in \mathbf{Q}, l_k \in \mathbf{B}_n, r \in \mathbf{S}\}.$$

Clearly, the restriction to $r = 1$, yields a free group, i.e., $\underline{\mathbf{W}}(\mathbf{B}_n, \{1\}) = \mathbf{W}_B$. The set is closed with respect to the multiplication of scaled labels, it is associative, and the inverse always exists, as the products of the label inverse and the scale inverse. It is also commutative. Thus, the generalized group $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ is an Abelian group over $\mathbf{Q}$ and its members are monomials of a group ring $\mathbf{R}[\mathbf{W}_B]$.

Some of the basic properties of the free groups are conserved in the extended structure. For example, there is the natural tower of subgroups and $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ is solvable. We proceed immediately to extend this structure to include names.

Let $\mathbf{B}_N$ be a finite set of labels we call "names". Let $\mathbf{N}$ be the Free Abelian multiplicative group over $\mathbf{Q}$ generated from the words in $\mathbf{B}_N$, whose only intersection with $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ is the unit label. This is guaranteed by having an empty intersection of the basis sets $\mathbf{B}_n$ and $\mathbf{B}_N$. Let, **name**: $\mathbf{N} \to \mathbf{W}(\mathbf{B}_n, \mathbf{S})$ be an injective homomorphism. This map is general not surjective. Then the names extended set theoretic product,

$$\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N}) = \mathbf{W}(\mathbf{B}_n, \mathbf{S})\mathbf{N} = \mathbf{N}\mathbf{W}(\mathbf{B}_n, \mathbf{S})$$

is an Abelian multiplicative group (essentially, by construction). At this point it is worth noting that no rational powers of the names appear. This is not a significant limitation. One can just as easily abandon the free group of names $\mathbf{N}$ and use the non-finitely generated group containing all rational powers of the names in $\mathbf{B}_N$ as well. This will allow us to consider, for example, $\mathrm{J}^{1/3}$, if we wish. This further extension is trivial.

3.3. Equivalence of labels

The *irreducible* form of a label is the expression, in terms of a reduced label, and possibly a scale factor r:

$$\textbf{Irreducibleform}[\text{label}] = r \prod_{k=1}^{n} l_k^{a_k} = r \ \text{label\_reduced}.$$

Two labels are *equivalent*, $\text{label}_1 == \text{label}_2$, iff the reduced label of their irreducible forms is the same. The map, **Irreducibleform**, from $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$ to $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ operates as follows:

$$\begin{aligned}\textbf{Irreducibleform}[\textbf{name}[r\ \text{label}]] &= \textbf{Irreducibleform}[r\ \text{label}] \\ &= r\ \textbf{IrreducibleForm}[\text{label}]\end{aligned}$$

and,

$$\textbf{IrreducibleForm}[\text{label\_reduced}] = \text{label\_reduced}$$

for any label in $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$. Note that according to this definition, all labels in $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ are reduced (or irreducible). The Irreducibleform map is surjective but not injective.

The utility of recognizing the concept of equivalence is that any label in $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$ can be replaced by another equivalent to it at our convenience. For example, the set $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$ contains labels (and factors in labels) of the form $\textbf{name}[r\ \text{label}]r^{-1}\ \text{label}^{-1} == \mathbf{1}$, i.e. equivalent to the unit label. In fact, the irreducible form of $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$ is just $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$. The addition of the set $\mathbf{N}$ is merely a convenience and does not change the system of units in the least. In practice, one generally reduces the labels after operations with q-numbers to the reduced form (that is, multiplying the numeric part of u by r), and perhaps abbreviating some of the factors by using names in $\mathbf{N}$.

Examples. **Irreducibleform**[newton kg^{-1} s^2] = m, is actually reduced.

$$\begin{aligned}\textbf{Irreducibleform}[\ \text{newton}\ \text{kg}^{-1}\ \text{s}^2] &= 10^{-3}\ \text{m} \\ &= \text{mm},\ \text{is not reduced, but scaled.}\end{aligned}$$

3.4. Particular realizations of $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$

The structure of $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ is specified by two aspects: (1) the algebraic structure imposed by the definitions above and (2) the nature of physical law.

The nature of physical law indicates that all labels can be generated from a small set of primary labels (the basis). The choice of these primary labels specifies a system of units, and thus the explicit form of the set **W**. The fundamental statement of physical law is that all processes in nature utilize as currency of interaction a quantity called energy, and that this quantity is remains the same in any interaction. Thus energy can be changed in form, as from mass into kinetic energy, or from electromagnetic into heat, or from heat into mechanical, etc., but the total amount remains constant in any process.

Physical processes are of two basic types: mechanical and entropic. The simplest specification of mechanical energy can be obtained by considering the simple kinetic energy of a free particle of mass m, moving with velocity v. Newtonian physics tells us that $E = mv^2/2$. Thus the specification of mechanical energy requires the specification of a scale for mass, a scale for length, and a scale for time. In addition, thermodynamics shows the existence of non-mechanical energy and leads us, via the second law, to the definition of temperature. This implies that we need a scale for temperature. In addition, three other base units are defined: an amount of substance, an amount of electric current, and an amount of luminous intensity [11].

The finite set of *primary labels*, {mass_scale, length_scale, time_scale, temperature_scale, substance_scale, current_scale, luminosity_scale}, can be called the *basis set* of the system of units.

Two other systems of units are in common use (but many others exist). The English or British system, used mostly in engineering, is defined by the basis set {pound, foot, second, Rankine, mol}. An older decimal system still in use by die-hard scientists, is the "cgs" system {gram, cm, second, Kelvin, mol}. Physical law demands that all systems of units produce the exact same specification of a physical system. This implies that there is a set of conversion factors between the systems, in the form of scaled labels. For example,

$$\begin{aligned} &\text{foot} = (3145/10000) \ \text{meter}, \\ &\text{pound} = (10/22) \ \text{kilogram}, \\ &\text{Rankine} = (9/5) \ \text{Kelvin}. \end{aligned}$$

Extensive conversion tables between units can be found in the *Handbook of Chemistry and Physics* [12]. Note that these examples of other unit systems are of lesser rank that the SI system, defined below. In the "CGS" system, for example, one can omit a scale for charge, and therefore express charge in terms of square roots of other scales. This has some inconveniences, but it is a possibility. Thus we retain our generality by allowing monomials over rational numbers.

For concreteness, we will give the explicit form of one set $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ for the SI (International System) system of units, $\mathbf{B}_7 =$ {kilogram, meter, second, Kelvin, mole, ampere, candela}, below. We note in passing that one can also work with other members of the tower of the set $\mathbf{W}$ by using a subset of the generating set. The minimal subset where energy can be defined is, for example, the "MKS" system defined by {kg, m, s} which is suitable for describing the behavior of a small set of particles with prescribed gravitational interactions.

In the case of the restricted rank 5 SI system where we omit current and luminosity for brevity, we have the generating set: {kg, m, s, Kelvin, mol}. The extension to the full set of rank 7, is obvious. Then,

$$\mathbf{W}(\mathbf{B}_5, \mathbf{S}) = \{l, l = r \ \text{kg}^a \ \text{m}^b \ \text{s}^c \ \text{Kelvin}^d \ \text{mol}^e; a, b, c, d, e \in \text{rationals}, r \in \mathbf{S}\}.$$

And the multiplicative group $\mathbf{S} = \{10^n, n \in \mathbf{Z}\}$. A restriction to a finite subgroup of this $\mathbf{S}$ would be quite sufficient in practice.

4. Algebra of q-numbers

Now that we have defined the algebra of the labels, we must define an algebra for the q-numbers themselves. Let **q** be the set of q-numbers. A quantity u is a q-number iff u is the ordered pair, $u = \{\mathrm{u}, \mathrm{label_u}\}$, where the numerical value $\mathrm{u} \in \mathbf{C}$, and $\mathrm{label_u} \in \mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$. Often u is restricted to the real numbers, **R**.

4.1. *Addition*

The addition of q-numbers is defined as,

$$v = u + w = \{\mathrm{u}, s\ \mathrm{label_u}\} + \{w, s'\ \mathrm{label}_w\} = \{us + ws',\ \mathrm{label}\}$$

iff for the *reduced* labels, $\mathrm{label_u} == \mathrm{label_w} == \mathrm{label}$. That it, the labels must be equivalent via the irreducible form map introduced in the previous section. Of course, after the result of the addition, we are free to rescale and rename if convenient. That is, we can write,

$$\begin{aligned} v = \{us + ws',\ \mathrm{label}\} &= \{(us + ws')/s'',\ s''\ \mathrm{label}\} \\ &= \{(us + ws')/s'',\ \mathrm{name}[s''\ \mathrm{label}]\}. \end{aligned}$$

Example. $10\,\mathrm{cm} + 1\ \mathrm{m} = 1.1\ \mathrm{m} = 110\,\mathrm{cm}$, where $\mathrm{cm} = \mathbf{name}[10^{-2}\ \mathrm{m}]$.

This is a severe restriction upon addition, inherited from the general lack of additivity of labels. Thus, as detailed below, the addition operation partitions the set **q** into disjoint *one-dimensional vector spaces*, one for each (reduced) label in $\mathbf{W}_B$.

Definition. Two q-numbers are congruent, iff their *reduced* labels are identical. We can say that the addition of q-numbers must be congruent, or is defined only between congruent q-numbers.

Multiplication by a complex number is distributive over congruent q-numbers:

$$\mathrm{z}(u + w) = \{zu + zw,\ \mathrm{label}\} = \mathrm{z}u + \mathrm{z}w.$$

4.2. Map to the complex (or real) numbers

For the set of all q-numbers congruent to v, there exists a special set of q-numbers in this set with a *reduced* label, which carry all congruent q-numbers to the pure number form:

$$N[\text{label}] = (1, \text{label}^{-1}).$$

Then, $v = \nu N[\text{label}] = N[\text{label}] \cdot \nu = \{v, \mathbf{1}\}$ is a pure number.

Proof. All quantities of the form $\{z, \mathbf{1}\}, z \in \mathbf{C}$, are congruent with each other and remain so under addition and multiplication. Thus, the map $\{z, \mathbf{1}\} \to z$ is an isomorphism. Thus these quantities have the same properties as the complex numbers and cannot be distinguished from them. □

If a q-number has a scaled label, it must first be transformed to a reduced label to arrive at the pure number form. Thus,

$$u\_s = \{u, s\,\text{label}\} \to \{us,\ \text{label}\} = us \text{ and,}$$
$$N[\text{label}] \cdot us = \{us, \mathbf{1}\} = us.$$

Magnitude: The magnitude of a q-number is the map:

$$\|\nu\| = \{|v|,\ \text{label}\}.$$

Multiplication: The quantity $\nu = uw$ is always a q-number.

$$\nu = \{\mathrm{u},\ \text{label}_\mathrm{u}\}\{w,\ \text{label}_\mathrm{w}\} = \{\mathrm{u}w,\ \text{label}_\mathrm{u}\,\text{label}_\mathrm{w}\}.$$

We note that, for non-unit labels, v is never congruent with the factors. The multiplicative inverse of the q-number v is also defined for non-zero magnitude $|v|$,

$$\nu^{-1} = \{1/v,\ \text{label}_v^{-1}\},$$

and the neutral element is just $\{1, \mathbf{1}\}$. Thus $\mathbf{q}$ $\backslash\{0, \mathbf{1}\}$ forms an Abelian group under multiplication, where we take $\{0, \mathbf{1}\}$ to be the equivalence class of all zero magnitude q-numbers.

4.3. Algebraic structure of q

We have been able to determine that $\mathbf{q}$, excluding the zero magnitude class, is a multiplicative Abelian group. A concise representation of $\mathbf{q}$ is

$$\mathbf{q}(\mathbf{C}, \mathbf{W}_B) = \mathbf{C} \times \mathbf{W}_B.$$

If we restrict ourselves to the vector space of q-numbers congruent to a given label, then it is obvious that the elements of this vector space (see below) do not form a multiplicative group. Thus, even though within the vector space we do have additivity, only one vector space contains the multiplicative unit label. Thus, in general we do not have a ring structure for the operations of addition and multiplication of q-numbers. There is only one trivial ring, for the vector space containing the unit label. This ring is that of the complex or real numbers (which are actually fields). The extensions $\mathbf{W}(\mathbf{B}_n, \mathbf{S})$ and $\mathbf{W}(\mathbf{B}_n, \mathbf{S}, \mathbf{N})$ extend $\mathbf{q}$ with scaling and convenient names.

4.4. Vector spaces

What kind of space does a general n-tuple of q-numbers $\{v_1, \ldots, v_n\}$ belong to? We can surely make maps where not all the q-numbers are congruent. However, if one restricts to a congruent n-tuple, then there is an induced vector space from the pure number components, and things are normal. This is what is actually meant in practice when we say, "momentum space", "coordinate space", "velocity space", etc. Physicists are careful to use only congruent members in defining these vector spaces.

Theorem. The set $\mathbf{V}_n$ of all n-tuples of congruent q-numbers $\nu = \{\nu_1, \ldots, \nu_n\}$ is an n-dimensional vector space. (This trivially covers the one-dimensional case alluded to above.)

Proof. (1) Abelian additive group:

Define the addition, of q-numbers as, $\boldsymbol{v} + \boldsymbol{u} = \{v_{1+}u_1, \ldots, v_n + u_n\}$, then we obtain:

$\boldsymbol{v} + \boldsymbol{u} = \{\nu_1 + u_1, \ldots, \nu_n + u_n\} \in \mathbf{V}_\mathrm{n}$, "closure"
$\boldsymbol{w} + (\boldsymbol{v} + \boldsymbol{u}) = (\boldsymbol{w} + \boldsymbol{v}) + \boldsymbol{u}$ "associativity"
$0 + \boldsymbol{v} = \boldsymbol{v} + 0 = \boldsymbol{v}$, $0 = \{0, \ldots., 0\}$, $\boldsymbol{v} \in \mathbf{V}_n$ "neutral element"
$\boldsymbol{v}^{-1} = -\boldsymbol{v} = \{-\boldsymbol{v}_1, \ldots, -\boldsymbol{v}_n\}$ "inverse element"
$\boldsymbol{v} + \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{v}$, "commutativity"

(2) Define the multiplication of an n-tuple by a complex number by,

$$z\boldsymbol{v} = \{z\boldsymbol{v}_1, \ldots, z\boldsymbol{v}_n\}, \text{ then,}$$

Since the $z \in \mathbf{X}$ form a field, such a multiplication is associative, distributive over the field, distributive over the q-n tuples:

$$\begin{aligned} z_1(z_2\boldsymbol{v}) &= \{z_1z_2\nu_1, \ldots, z_1z_2\nu_n\} = (z_1z_2)\boldsymbol{v} \\ (z_{1+}z_2)\boldsymbol{v} &= \{(z_1 + z_2)\nu_1, \ldots, (z_1 + z_2)\nu_n\} = z_1\boldsymbol{v} + z_2\boldsymbol{v}, \\ z(\boldsymbol{v}_1 + \boldsymbol{v}_2) &= \{z(\nu_1 + u_1), \ldots, z(\nu_n + u_n)\} = z\boldsymbol{v}_1 + z\boldsymbol{v}_2. \end{aligned}$$

$\mathbf{V}_n$ is a normed vector space, for the Euclidian norm is well defined:

$$\|\boldsymbol{v}\|^2 = \sum_{k=1}^{n} |\nu_k|^2, \quad \text{where } |\nu_k|^2 = \nu_k^* \ \nu_k = \{\nu_k^*, \text{label}\}\{\upsilon_k, \text{label}\} = \{|\upsilon_k|^2, \ \text{label}^2\}$$

and the conjugate vector $\boldsymbol{v}^* \in \mathbf{V}_n$ is used to make the inner product, $\langle \boldsymbol{v}^* | \boldsymbol{v} \rangle = \|v\|^2$. The norm is then $\|\boldsymbol{v}\| = \left\{\sqrt{\sum_{k=1}^{n} |\upsilon_k|^2}, \ \text{label}\right\}$, and is congruent with members of $\mathbf{V}_n$.

4.5. *The group ring* $\mathbf{C}[\mathbf{W}_B]$

Do the q-numbers fit into a larger algebraic structure? We have remarked previously that a q-number, as usually written, u label, can be viewed as "pure monomials" in a Group Ring [10] over the complex (or real) field. The Group Ring is the set of all possible formal sums:

$$c_1 l(a_{11}, \ldots, a_{1n}) \& \cdots \& c_k l(a_{k1}, \ldots, a_{kn}),$$

where $c_k \in \mathbf{C}$, the addition, & is associative and we identify the product of the multiplicative identity in the field times a label, as just the label. It is clear that $\mathbf{W}_B$ is contained in the Group Ring. Since we have emphasized that neither $\mathbf{q}$ nor $\mathbf{W}_B$ is a ring, how can this be interpreted? Since q-numbers convey information about the properties of specific systems found in nature, it is natural to think of all the properties that a given system could have. These properties will form a list of q-numbers, and they will each have their own numerical value and label. Thus we can identify the set of properties of a particular system with an element in the Group Ring because these properties exist at once. Different systems will have different known properties and will correspond to different elements in the Group Ring. The entire Group Ring can be interpreted as the set of all possible properties of all possible systems in nature. Physical theory, in principle, provides a link between all the monomials because they can all be obtained from a small set of fundamental q-numbers in nature. Thus, physical theory is a map between a set such as {number of electrons, number of protons, number of neutrons, speed of light, Planck's constant, etc.} $\rightarrow \mathbf{C}[\mathbf{W}_B]$. The more fundamental the theory, the smaller the domain, and the larger the range of the map. We note in passing, that in the quantum domain (where not all systems are distinguishable), this map is not even injective, but it is so in the macroscopic domain.

Furthermore, physical theory can, from time to time in the course of its development, produce identifications between certain monomials in the Group Ring. A very famous example is Einstein's identification of energy $E = \mathrm{E\,J}$ with mass, via $E = mc^2$. Thus, two monomials that appeared separately in the Group Ring, can become identified. This implies, that in general, there exists an ideal of

$\mathbf{W}_B$ such that the quotient group is all that needs to be considered. Our knowledge of the Ideal is imperfect, and will be so into the future.

5. Applications

That there exist applications is obvious, but in this section we emphasize the concepts of well-posed expressions and the consequent utility of dimensional analysis.

5.1. *Expressions containing q-numbers*

There are two basic types of expressions involving q-numbers. The first type is a map from q-numbers to the complex numbers, while the second involves a map from q-numbers to a q-number.

Definition. Well-posed expressions.

An expression $f == g[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$ is well posed iff f and g are congruent.

If $\text{label}_f = \text{label}_g = \mathbf{1}$, then the expression is a map from a set of q-numbers to the complex numbers.

Linear map. $f == \Sigma_k a_k \boldsymbol{v}_k$ requires the existence of a set of q-numbers $a_k = \{a_k, \text{label}_{vk}^{-1}\}$, such that the products $a_k \boldsymbol{v}_k$ have the unit label.

Non-linear map. The map must be made of products and sufficient auxiliary q-numbers so that the label of each complete assembly of factors is equivalent to $\mathbf{1}$.

The preceeding can be generalized to a map from q-numbers to a q-number. In the linear map case, then $a_k = \{\mathrm{a}_k, \text{label}_f\, \text{label}_{vk}^{-1}\}$ and in the non-linear case, each complete assembly of factors must have a label equivalent to label_f. These statements presage Buckingham's theorem, the basic theorem of dimensional analysis see below.

Corollary. The arguments of functions of pure numbers must be well posed, that is, congruent to a pure number. Thus, the argument of a pure function must always be of the form av, accompanied by an auxiliary q-number $a = \{\mathrm{a}, \text{label}_v^{-1}\}$.

Example. If x is a q-number, then the argument of cos must be $\cos[ax]$. Thus it is not well posed to say, "let t be the *time*, then an oscillatory behavior can be represented by $\cos[t]$"; one must say, instead, $\cos[\omega t]$. Thus, the frequency $\omega = \{\omega, \text{Hz} = \text{sec}^{-1}\}$ must appear because *time* is a q-number, $t = \{t, \text{sec}\}$.

5.2. *Remark: measurement*

The results of any measurement are q-numbers over the real numbers. Thus, $\mathrm{Re}[z] = \{\mathrm{Re}[z], \mathrm{label}\}$ and $\mathrm{Im}[v] = \{\mathrm{Im}[v], \mathrm{label}\}$ can be associated with results of measurements.

Example. The dielectric constant of an isotropic material is of the form,

$$E = E_o(a + \mathrm{i}b)$$

with $\mathrm{label} = \mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$ and E_o a is related to the electrical energy stored in a material, while E_o b is related to the energy loss to heat in an insulator or to the conductivity in a conductor.

5.3. *Operators*

Higher-order quantities such as tensors and operators can be constructed strictly out of congruent q-numbers. For example, a derivative with respect to a q-number x is defined as

$$\frac{\partial}{\partial x} = \left\{\frac{\partial}{\partial x}, \mathrm{label}_x^{-1}\right\}.$$

In order to satisfy congruency, auxiliary q-numbers will often be introduced. These are usually called parameters. Examples of these are the fundamental constants of nature (speed of light, Planck's constant, the masses and charges of the elementary particles and atoms, etc.), or parameters of oscillations such as the frequency and wavelength. In the following, we give some explicit examples of some of these constructs.

5.4. *Dimensionless variables*

In order to obtain the numeric solution to a physical problem, one must find a congruent label for all groupings of variables, and then map to a pure number using the appropriate N[label] function. The last step can be done implicitly or explicitly. Mapping to a pure number is often referred to as introducing dimensionless variables.

Implicit example: Ideal gas law.

The relation between q-numbers is $P == nRT/V$, where P is the pressure, n is the number of moles, R is the gas constant (a parameter), T is the temperature, and V the volume of the vessel containing the gas. The explicit form of these q-numbers, in standard units (not abbreviated, for clarity), is

$$P = \{\mathrm{P},\ \mathrm{Pascal}\},\quad T = \{\mathrm{T},\ \mathrm{Kelvin}\},\quad V = \{\mathrm{V},\ \mathrm{liter}\},\quad n = \{\mathrm{n},\ \mathrm{mole}\},$$

$R = \{\mathrm{R},\ \mathrm{Joule\,Kelvin}^{-1}\ \mathrm{mole}^{-1}\}$.

A naïve computation using these standard units would lead the unwary student to assume that the label Pascal = = Joule/liter, which is incorrect, for there is a hidden scale factor. One does not discover this unless one works out the algebra for the labels, and this is the point of the example. A partially reduced form of Pascal = newton m^{-2}, while liter = $10^{-3}\,\mathrm{m}^3$, and Joule = newton m. Note that it is not necessary to go all the way to the irreducible form of the label –*a* "common denominator" will do. Thus, we see that by introducing the auxiliary q-number, or conversion factor, $r = \{10^{-3}\,\mathrm{m}^3\,\mathrm{liter}^{-1}\}$ we can achieve a well posed expression, given the assumed labels on the q-numbers at hand. The explicit computation now is

$$\begin{aligned}\{\mathrm{P},\ \mathrm{Pascal}\} = {} & \{\mathrm{n},\ \mathrm{mol}\}\{\mathrm{R},\ \mathrm{Joule\,Kelvin}^{-1}\ \mathrm{mol}^{-1}\}\\ & \times\{\mathrm{T},\ \mathrm{Kelvin}\}/(\{10^{-3},\mathrm{m}^3\,\mathrm{liter}^{-1}\}\{\mathrm{V},\ \mathrm{liter}\}).\end{aligned}$$

Given this well posed expression, the student can now use the calculator on the purely numeric quantities n, R, T, V and 10^{-3}, to get the correct value for P. The student must report a q-number, not just the numeric quantity P. The correct answer is: $P = \mathrm{P\,Pascal}$, using the standard nomenclature that omits the curly brackets {, }.

Explicit example. The harmonic oscillator Schrödinger equation.

The Hamiltonian operator for the Harmonic oscillator problem is given by [13],

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$$

and the Schrödinger equation is the eigenvalue problem $H\Psi(x = E\Psi(x)$. The explicit forms of the quantities are: $H = \{\mathrm{H},\ \mathrm{Joule}\}$, $x = \{\mathrm{x},\ \mathrm{m}\}$, $m = \{\mathrm{m}',\mathrm{kg}\}$, $\hbar = \{\hbar,\ \mathrm{Joule\,s}\}$, $k = \{\mathrm{k},\mathrm{Joule\,m}^{-2}\}$, $E = \{\mathrm{E},\ \mathrm{Joule}\}$. The irreducible form of Joule = kg m^2 s^{-2} and this shows that the derivative term is congruent to Joule and the expression is well posed. To obtain the numerical or analytical solution of this equation, it is convenient to map it to a pure number form by the explicit introduction of dimensionless variables. Let the auxiliary q-number $\alpha^2 = mk/\hbar^2$ and multiply the entire equation by $-2\alpha^2/k$. Then the naked derivative term is congruent to m^{-2} and so must $-\alpha^2x^2$ be. This shows that $\alpha^2 = \{\alpha^2, m^{-4}\}$ and that $y = \sqrt{\alpha}x$ is a pure number, or dimensionless variable. Now divide the previous form of the equation by α to obtain the equation in pure number form:

$$\left(\frac{\partial^2}{\partial y^2} - y^2 + \beta\right)\psi(y) = 0,$$

where the eigenvalue is $\beta = 2mkE/\hbar^2$, $\alpha = 2E/\hbar\omega$ and the classical frequency is $\omega^2 = k/m$, $(\omega = \{\omega,\mathrm{s}^{-1}\})$. The analytical solution shows that the eigenfunctions are expressible as a product of a Hermite polynomial of order n, times a Gaussian function and the eigenvalule is just $n+1$. The final solution (normalized by N_n) in terms of q-numbers is:

$$E_n = \hbar\omega(n+1/2), \quad n = 0, 1, 2, 3, \dots ,$$
$$\psi(\sqrt{\alpha}x) = N_n H_n(\sqrt{\alpha}x)\exp(-\alpha x^2).$$

5.5. Dimensional analysis

Equations that describe the natural world must be well-posed expressions between q-numbers. Thus, when one does not know before hand what the specific form of the governing relations are in a specific circumstance, one can use the well posed requirement to obtain the general form of the dependence on the variables if a complete set of these is at hand [3].

A simple example is the following. Suppose that we have a particle of mass m moving with velocity v in free space. We would like to know what the energy of such a particle is. Since it is in free space, there are no other variables to consider, thus the expression for the energy must be made from just these two q-numbers. Since energy is congruent to Joule $=$ $\mathrm{kg\,m^2\,s^{-2}}$, then it follows immediately that $E = bmv^2$, where b is an unknown scale factor. If one does a detailed Newtonian analysis, one finds that the scale factor $b = 1/2$.

This exemplifies the power of dimensional analysis. The fact that the q-number structure can be given a formal algebraic basis establishes that dimensional analysis is a sound mathematical procedure when the set of complete variables can be unequivocally chosen. This is not always the case. One may find, for example, that one needs a length scale to complete the congruence in a particular problem, and that there is more than one length scale to choose from. In that case, however, some experimental observation can help to choose the correct length scale.

5.6. Buckingham's theorem

The fact that we cannot add quantities that contain different labels leads to a rigorous method of determining the possible relations that may exist among a set of arbitrary q-numbers. The algebraic method of obtaining these relations is based on Buckingham's theorem or the "Π" theorem [14]. Langhaar [2] has presented a complete demonstration of this theorem utilizing linear algebra. We extend the theorem to the case of rational exponents here.

In order to state and prove this theorem, we return to the map between an element, $l(a_1, \dots, a_n) = l(a_1, \dots, a_n; 0)$, of $\mathbf{W}(n)$ and the n-tuple of exponents, $(a_1, \dots, a_n)$. It is evident that the set of exponents, $E(n)$, found in $\mathbf{W}(n)$

form an Abelian additive group, where the neutral element is $(0, 0, \ldots, 0)$, and the inverse is $(-a_1, \ldots, -a_n)$. Furthermore, the map $\hbar(a_1, \ldots, a_n) \to (a_1, \ldots, a_n)$ is an isomorphism between $\mathbf{W}(n)$ and $E(n)$. This map is clearly bijective and it preserves the group multiplication table:

$$l(a_1, \ldots, a_n) \cdot l(a'_1, \ldots, a'_n) = l(a_1 + a'_1, \ldots, a_n + a'_n) \to \\ (a_1, \ldots, a_n) + (a'_1, \ldots, a'_n) = (a_1 + a'_1, \ldots, a_n + a'_n).$$

Let $(e_1, \ldots, e_n)$ be a basis for $E(n)$. Then the exponent a_k of $\hbar_k$ of any q-number is a linear combination of the basis, that is, $a_k = \sum_{j=1}^{n} \alpha_{jk} e_j$. The elements of the matrix α are rational numbers.

Now, consider an arbitrary set of q-numbers, $q_1, \ldots, q_\nu$ in $\mathbf{W}(n)$. We need to construct monomials with integer powers of these q-numbers that are congruent to the unit label. The question is, how many can we form, and what are their exponents?

An arbitrary q-number has rational powers in the label. Thus, the first thing we do is to find the smallest integer that is divisible by all the denominators of the exponents in label$_\mu$. Call that integer n_μ. Then, (label$_\mu$)$^{n_\mu}$ has only integer powers of the basis labels. Now we must form monomials,

$$x = \sum_{j=1}^{\nu} {q_j}^{n_j \rho_j}, \text{ where } \rho_j \text{ are integers, and } x \text{ is congruent to the unit label.}$$

The exponent of label$_x$ in $E(n)$ is given by, $\sum_{k=1}^{\nu} n_k \rho_k \sum_{j=1}^{n} \alpha_{jk} e_j$ and this will yield a zero exponent only if the coefficient of each basis element is zero: $\sum_{k=1}^{\nu} n_k \alpha_{jk} \rho_k = 0$.

This is a system of linear equations which will have m linearly independent solutions for the exponents ρ_k, where $m \leqslant n < \nu$ is the rank of the matrix α_{jk}. The matrix $n_k \alpha_{jk}$ has integer entries. Thus, we have demonstrated that there are $\nu - m$ independent monomials that can be made up of the arbitrary set of q-numbers; call them x_k. Then, it follows at once that there exists a function, $f(x_1, \ldots, x_{\nu-m}) = c$, a constant, that relates all of these monomials. This is essentially the content of Buckingham's theorem. We have generalized the proof to consider the possibility of rational powers in the q-numbers, thereby introducing the auxiliary integers n_k into the picture. Examples of the solution of the system of equations for particular systems can be found in the cited books on dimensional analysis [1–4].

6. Summary

We have presented a general algebraic basis for arbitrary systems of units such as those used in physical sciences, engineering, and economics. The algebraic structure of the infinite sets of labels has been established: such sets $\mathbf{W}_B$

are infinite Abelian multiplicative groups with a finite basis. $\mathbf{W}_B$ is solvable as it admits a tower of subgroups. Extensions to include the possibility of rational powers of labels have been included, as well as the addition of named labels. Named labels are an essential feature of all practical systems of units. Furthermore, we have shown that physical quantities are q-numbers $\in \mathbf{q} = \mathbf{C} \times \mathbf{W}_B \cdot \mathbf{q}$ is an Abelian multiplicative group, and it is not a ring. $\mathbf{q}$ admits decomposition into one-dimensional normed vector spaces over the field $\mathbf{C}$ among members with equivalent labels. These properties lead naturally to the concept of well posed relations, and to Buckingham's theorem of dimensional analysis.

Acknowledgments

The author wishes to acknowledge insightful discussions with Prof. Joseph Gubeladze, Mathematics, SFSU.

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